What is Mathematics?

Some Personal Reflections

by P. Y. H. Pang

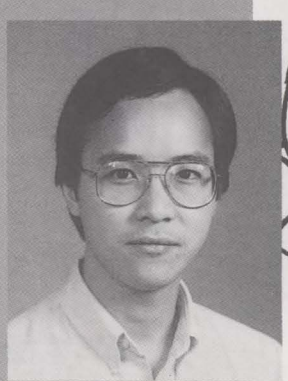
So we have this dilemma . . .

Mathematics is mathematics because it is difficult to relate to . . .

What is the point of a point?

Graduates of mathematics are much sought after . . .

Dr Peter Pang Yu Hin obtained his BSc from Toronto, Canada, and PhD from the University of Illinois at Urbana Champaign. He has been a lecturer at the Department of Mathematics at NUS since 1988. His research interest includes geometry, topology and homotopy theory.



So we have this dilemma . . .

Most people hate mathematics, there is no question about it. But then most people also hate cod liver oil. What they do, regarding cod liver oil, is that they stop taking it when they wield sufficient power, which can always be achieved. (And when they have their own children, they force them to take it, of course.) Mathematics is quite a different matter. It seems that the more power you wield, the more it affects you. You have to hate it for the rest of your life. Few people can really escape mathematics, it is everywhere. Sure, if you are going to be any sort of a professional, you cannot really avoid coming into contact with some kind of mathematics. Perhaps you have to come to some sort of understanding of schedule or inventory control as a manager, or you have to have some very basic understanding of statistics as an investment consultant or a medical practitioner. So we have this dilemma. On the one hand, we probably all agree that mathematics is useful, even though we may not be quite sure in exactly what way. On the other hand, many of us find mathematics unappealing, we find it difficult to relate to, and we fail to see the direct relevance of the concepts and formulae of mathematics to the real world. To too many of us, mathematics means self-denial and suffering (some people actually believe that this is why everyone must learn mathematics!), and we question whether all this suffering is necessary.

I suppose one reason that people do not take to mathematics is that it is more like school work than other school work. Unlike English composition, where you just have to express your "normal" opinions, in mathematics you are always given really ridiculous questions, like " Mr. Wong is two years older than the sum of the ages of his three sons, blah, blah, blah. What is the age of the youngest son?" Well, if you want to know the age, why don't you just go and ask him? Or, "You bought 0.7328 kg of spinach which cost $2.99 per kg, blah, blah, blah." I hate spinach and will never buy it! Also, you are so easily put off by teachers who require you to answer the question "How many pieces of cake has John left?" with "John has 39 pieces of cake left", not one word more, not one word less. Mathematics can be awfully dry if not put in the right perspective.

Mathematics is mathematics because it is difficult to relate to . . .

Mathematics is mathematics because it is difficult to relate to. This certainly is a controversial statement. Let me explain. Mathematics is different from the other sciences in that it is not based on empiricism. There are no experiments in the usual sense in mathematics. The theorems in mathematics are true once they are proven, they need not be checked against empirical data and be fine-tuned by further experimentation. In fact, the objects of mathematics need not have any physical meaning, and the operations allowed on them need not correspond to anything physically enactable. To put it controversially, mathematics is a game. A game in which you set the rules, subject to a very general "universally accepted" code of logic. Mathematicians talk about the fourth dimension without the slightest qualms. I quote Dr. N. J. A. Sloane: "There has been a great deal of nonsense written in science fiction and elsewhere about the mysteries of the fourth dimension. In mathematical discussions one must not assume, as the physicist does, that the fourth dimension represents time. In mathematics four-dimensional space (simply) consists of points with four coordinates instead of three (and the same holds for any number of dimensions)." They also have no feeling of guilt when they add in base two and come up with 1 + 1 = 10. Mathematics, in short, is the working of the human mind, pure and abstract. But then why is mathematics so useful? Why can it offer us so much insight into the problems of real-life?

What is the point of a point?

If you really think about it, the idea of a point (a geometric point) is an abstract idea. Have you ever seen a real point physically? (Remember that a point is an object of zero dimension.) But then also if you think about it, the idea of a number, say the number 1, is an abstract idea. Of course we can tell when there is only one ball, or when there is only one car, or when there is only one person. But coming up with the idea of numbers to describe quantity precisely, that is a real achievement. And this is what mathematics is all about. We have certain absolutely unavoidable problems, like the description of quantity and the understanding of basic geometric and mechanical concepts. In coming to terms with such intellectual challenges throughout the evolution of the human race,

More boys or girls?

Imagine a country has a law as follows: A couple must continue to bear children until they have a boy and they must stop bearing once they have a boy. Assuming no multiple births and infertility problems, and that a couple is equally likely to bear a boy or a girl, do you think that the country will likely to have more males or females?

we (the human species) have come up with the abstract concepts of numbers and points. We describe quantity in a precise way by counting, using numbers. We describe basic geometric and mechanical notions using points. A circle is the set of points equidistant from a fix point. A pendulum is a point mass of a fixed distance from a fixed point. Note that even though a real pendulum is not really a point mass, but a blob mass, its mechanics can be described satisfactorily by treating it as a point mass. Thus, mathematics is the way to solve problems by abstracting them and extracting from them certain quantitative and logical essentials. From this point of view, mathematics must be abstract, there are no two ways about it. And because it is abstract, it is easy to lose sight of the sometimes obscure physical reality that it represents, making it hard to relate to. In time, mathematics as an intellectual activity takes on a life of its own. It has, as we have said, become a game, the mirror reflecting the supremely sophisticated working of the human mind. But ultimately, what distinguishes good mathematics from bad mathematics is how closely it represents physical reality and, to a certain degree, aesthetics.

Our ordinary usage of language is imprecise. We may say: when it rains, I carry an umbrella. Do we mean: "when I can see that it is raining just before I go out, I will take an umbrella"?, or "whenever it rains, you will see me with an umbrella"? Mathematics addresses the problem of such ambiguities by developing a language of exactitude. You must say exactly what you mean before you can have any meaningful logically scrutiny of your ideas. Mathematical training gives you a logical and systematic mind and the aptitude for problem-solving. Perhaps this is why Plato put up the sign "No one ill-versed in geometry should enter" over his famous Academy more than two millennia ago.

Mathematics is a game through which we study the universe, physical or otherwise. Mathematics is a science because it is fundamentally quantitative. Mathematics is an art because good mathematics must be beautiful. Mathematics is about pushing the frontiers of human knowledge by abstraction, it is about searching for new ways and new points of view to represent our physical reality. At the most fundamental level, mathematics is about logical thinking.

Contrary to popular belief, mathematics is definitely not about speed. Difficult and important things are not supposed to be done in haste. Let me relate a story I heard from Professor Béla Bollobás. At Cambridge, there was a famous professor of mathematics named John Conway. When Rubik's cube first came out, he became interested in it and sat down and solved the puzzle completely. After some practice, he developed sufficient dexterity to perform the solution in just over a minute. He was proud of his achievement. At about that time, Rubik's cube competitions were held and televised, and the kids who took part in them could outperform him several times. Now these kids of course had very little understanding of the intricacies of the puzzle, it was just a matter of memorizing the algorithm and developing speed in the hand. To them, Rubik's cube was nothing of the intellectual game that it was to Conway.

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The power of mathematics is in its abstractness. Because it is abstract, it has few constraints and a wide range of applicability. Mathematics deals with the essence of everyday problems, whether in management or medicine. The methods of mathematics, or perhaps more importantly, the attitude developed in mathematical training, namely, the willingness to assimilate and apply abstract ideas, i.e., having an open mind, and the aptitude for inventive and systematic problem-solving, are indeed important assets in this highly competitive world. In fact, graduates of mathematics are much sought after because they are reputed to be fast and eager learners, and creative doers.

So to go back to our question: is the suffering really necessary? Nothing worthwhile does not require hard work, not in art, not in sports, and certainly not an intellectual games as challenging as mathematics. One does not become a chess grandmaster overnight, it takes years and years of boring drills and practice. But please note that though drilling is often necessary, it should not be blind drilling, drilling without a purpose. The large picture must not be lost. We benefit from mathematics because it trains our minds. It is unlike history and geography, where we learn mostly factual information. The formulae and theorems of mathematics are quickly forgotten, but the logic and problem-solving techniques are not, they are hard-wired in our minds, to be taken for granted for the rest of our lives. If some of the mathematical notions we have learnt before turn out to be useful later on in our career, which will most likely be true, it is just an added bonus, a secondary benefit, even though it may have a primary motive for learning mathematics. M^2